The Fixed-Mesh ALE method for Fluid-Structure Interaction problems

Ramon Codina and Joan Baiges

Technical University of Catalonia

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1. Introduction

2. Dirichlet boundary conditions in embedded grids
   - Strong imposition of boundary conditions
   - Weak imposition of boundary conditions

3. The Fixed-Mesh ALE method applied to Fluid-Structure Interaction problems

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1 Introduction

2 Dirichlet boundary conditions in embedded grids
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3 The Fixed-Mesh ALE method applied to Fluid-Structure Interaction problems

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Introduction

Objectives

We want to deal with multiphysics problems in time evolving domains.

Some of the necessary ingredients are:

- A method for imposing Dirichlet boundary conditions
- A strategy for dealing with the domain movement and computing time derivatives
ALE strategies are the usual way of dealing with moving domains:

**Features**
- At each time step the mesh is deformed following the domain movement.
- The finite element equations are written in a frame of reference attached to the mesh.

**Drawbacks**
- On very large deformations remeshing is needed.
- This can be time consuming and require of external software.
Fixed-Mesh methods appear to avoid remeshing.

- Ease of grid generation
- Ease of dealing with domain motion
- Usual approach: Cartesian grid methods

Features that distinguish a fixed mesh method

- Approximate imposition of Dirichlet boundary conditions
- Definition of the computational domain
- Newly created nodes and time derivative computation
- Handling of discontinuities of gradients and unknowns across the domain boundary
1. Introduction

2. Dirichlet boundary conditions in embedded grids
   - Strong imposition of boundary conditions
   - Weak imposition of boundary conditions

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Objectives

We want our method to fulfill:

- No need of extra degrees of freedom
- No need of using large penalty parameters
- Optimal order of accuracy should be attained
- Should be capable of dealing with flow problems

We propose:

- Strong imposition of boundary conditions in embedded grids
- Weak imposition of boundary conditions in embedded grids
Boundary value problem:

\[ \mathcal{L} u := -k \Delta u + a \cdot \nabla u + su = f \quad \text{in } \Omega \]

\[ u = \bar{u} \quad \text{on } \Gamma = \partial \Omega \]

where \( k > 0 \), \( a \) is the advection velocity, \( s \geq 0 \).
Setting

Boundary value problem:

\[ \mathcal{L}u := -k\Delta u + \mathbf{a} \cdot \nabla u + su = f \quad \text{in } \Omega \]

\[ u = \bar{u} \quad \text{on } \Gamma = \partial \Omega \]

where \( k > 0 \), \( \mathbf{a} \) is the advection velocity, \( s \geq 0 \).

\[ B(u_h, v_h) = k(\nabla u_h, \nabla v_h) + (\mathbf{a} \cdot \nabla u_h, v_h) + s(u_h, v_h) - k\langle \partial_n u_h, v_h \rangle \Gamma \]

Nitsche’s method: Find \( u_h \in V_h \) such that

\[ B(u_h, v_h) - k\langle u_h, \partial_n v_h \rangle \Gamma + \frac{\alpha k^*}{h} \langle u_h, v_h \rangle \Gamma \]

\[ = \langle f, v_h \rangle_{\Omega} - k\langle \bar{u}, \partial_n v_h \rangle \Gamma + \frac{\alpha k^*}{h} \langle \bar{u}, v_h \rangle \Gamma \]
Splitting Nitsche’s method

Consider the splitting $V_h = V_{h,0} \oplus V_{h,\Gamma}$, where $V_{h,0}$ is the subspace of $V_h$ of functions vanishing at the nodes outside $\Omega_{in}$, including its boundary, and $V_{h,\Gamma}$ the complement. Nitsche’s method can be obtained from:

\[
B(u_{h,0}, v_{h,0}) - k \langle \partial_n u_{h,0}, v_{h,0} \rangle_{\Gamma} + B(u_{h,\Gamma}, v_{h,0}) - k \langle \partial_n u_{h,\Gamma}, v_{h,0} \rangle_{\Gamma} = \langle f, v_{h,0} \rangle_{\Omega}
\]

\[
B(u_{h,0}, v_{h,\Gamma}) - k \langle \partial_n u_{h,0}, v_{h,\Gamma} \rangle_{\Gamma} + B(u_{h,\Gamma}, v_{h,\Gamma}) - k \langle \partial_n u_{h,\Gamma}, v_{h,\Gamma} \rangle_{\Gamma} = \langle f, v_{h,\Gamma} \rangle_{\Omega}
\]

\[
- k \langle \partial_n v_{h,0}, u_{h,0} \rangle_{\Gamma} - k \langle \partial_n v_{h,0}, u_{h,\Gamma} \rangle_{\Gamma} = -k \langle \partial_n v_{h,0}, \bar{u} \rangle_{\Gamma}
\]

\[
- k \langle \partial_n v_{h,\Gamma}, u_{h,0} \rangle_{\Gamma} - k \langle \partial_n v_{h,\Gamma}, u_{h,\Gamma} \rangle_{\Gamma} = -k \langle \partial_n v_{h,\Gamma}, \bar{u} \rangle_{\Gamma}
\]

\[
\frac{\alpha k^*}{h} \langle u_{h,0}, v_{h,0} \rangle_{\Gamma} + \frac{\alpha k^*}{h} \langle u_{h,\Gamma}, v_{h,0} \rangle_{\Gamma} = \frac{\alpha k^*}{h} \langle \bar{u}, v_{h,0} \rangle_{\Gamma}
\]

\[
\frac{\alpha k^*}{h} \langle u_{h,0}, v_{h,\Gamma} \rangle_{\Gamma} + \frac{\alpha k^*}{h} \langle u_{h,\Gamma}, v_{h,\Gamma} \rangle_{\Gamma} = \frac{\alpha k^*}{h} \langle \bar{u}, v_{h,\Gamma} \rangle_{\Gamma}
\]

We are using the degrees of freedom associated to external nodes in order to minimize $\|u_h - \bar{u}\|_{L^2(\Gamma)}$. 
Consider the splitting \( V_h = V_{h,0} \oplus V_{h,\Gamma} \), where \( V_{h,0} \) is the subspace of \( V_h \) of functions vanishing at the nodes outside \( \Omega_{\text{in}} \), including its boundary, and \( V_{h,\Gamma} \) the complement. Nitsche’s method can be obtained from:

\[
B(u_{h,0}, v_{h,0}) - k \langle \partial_n u_{h,0}, v_{h,0} \rangle_{\Gamma} + B(u_{h,\Gamma}, v_{h,0}) - k \langle \partial_n u_{h,\Gamma}, v_{h,0} \rangle_{\Gamma} = \langle f, v_{h,0} \rangle_{\Omega} \\
B(u_{h,0}, v_{h,\Gamma}) - k \langle \partial_n u_{h,0}, v_{h,\Gamma} \rangle_{\Gamma} + B(u_{h,\Gamma}, v_{h,\Gamma}) - k \langle \partial_n u_{h,\Gamma}, v_{h,\Gamma} \rangle_{\Gamma} = \langle f, v_{h,\Gamma} \rangle_{\Omega} \\
- k \langle \partial_n v_{h,0}, u_{h,0} \rangle_{\Gamma} - k \langle \partial_n v_{h,0}, u_{h,\Gamma} \rangle_{\Gamma} = -k \langle \partial_n v_{h,0}, \bar{u} \rangle_{\Gamma} \\
- k \langle \partial_n v_{h,\Gamma}, u_{h,0} \rangle_{\Gamma} - k \langle \partial_n v_{h,\Gamma}, u_{h,\Gamma} \rangle_{\Gamma} = -k \langle \partial_n v_{h,\Gamma}, \bar{u} \rangle_{\Gamma} \\
\frac{\alpha k^*}{h} \langle u_{h,0}, v_{h,0} \rangle_{\Gamma} + \frac{\alpha k^*}{h} \langle u_{h,\Gamma}, v_{h,0} \rangle_{\Gamma} = \frac{\alpha k^*}{h} \langle \bar{u}, v_{h,0} \rangle_{\Gamma} \\
\frac{\alpha k^*}{h} \langle u_{h,0}, v_{h,\Gamma} \rangle_{\Gamma} + \frac{\alpha k^*}{h} \langle u_{h,\Gamma}, v_{h,\Gamma} \rangle_{\Gamma} = \frac{\alpha k^*}{h} \langle \bar{u}, v_{h,\Gamma} \rangle_{\Gamma}
\]

We are using the degrees of freedom associated to external nodes in order to minimize \( ||u_h - \bar{u}||_{L^2(\Gamma)} \).
The proposed method

Find $u_{h,0} \in V_{h,0}$ and $u_{h,\Gamma} \in V_{h,\Gamma}$ such that

$$B(u_{h,0}, v_{h,0}) + B(u_{h,\Gamma}, v_{h,0}) - k\langle \partial_n u_{h,0}, v_{h,0} \rangle_{\Gamma} - k\langle \partial_n u_{h,\Gamma}, v_{h,0} \rangle_{\Gamma} = \langle f, v_{h,0} \rangle_{\Omega}$$

$$\frac{\alpha k^*}{h} \langle u_{h,0}, v_{h,\Gamma} \rangle_{\Gamma} + \frac{\alpha k^*}{h} \langle u_{h,\Gamma}, v_{h,\Gamma} \rangle_{\Gamma} = \frac{\alpha k^*}{h} \langle \vec{u}, v_{h,\Gamma} \rangle_{\Gamma}$$

for all $v_{h,0} \in V_{h,0}$ and $v_{h,\Gamma} \in V_{h,\Gamma}$.

Properties

- When $\Gamma$ coincides with $\partial \Omega_h$, the boundary condition is imposed exactly (provided $\vec{u}$ is a finite element function).
- There are no parameters to be tuned ($\frac{\alpha k^*}{h}$ can be dropped).
- The method is non-symmetric, even if $B$ is symmetric.

Theorem (Stability)

If $\Gamma$ is kept away from $\partial \Omega_{in}$, the formulation proposed is stable. As a consequence, the discrete problem admits a unique solution.
Dirichlet boundary conditions in embedded grids

Numerical examples

Results for Poisson’s problem

Left: convergence in $L^2(\Omega)$. Right: convergence in $L^2(\partial\Omega)$. 

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Dirichlet boundary conditions in embedded grids

Numerical examples

Results for the convection-diffusion equation
Conclusions

- Main idea: to use as degrees of freedom for the imposition of BCs those associated to the nodes adjacent to the boundary of the computational domain.
- Way to obtain the degrees of freedom: by minimizing the distance of the unknown to the boundary datum in the $L^2$ norm of the boundary.
- Final method: accurate and robust.
- Advantages:
  - No additional degrees of freedom.
  - No parameters to tune.
  - Easy extension to moving domains.
Two field formulation for Poisson’s problem:

\[-k \Delta u = f \quad \text{in } \Omega\]
\[\frac{1}{k} \sigma = \nabla u \quad \text{in } \Omega\]
\[u = \bar{u} \quad \text{on } \Gamma = \partial \Omega\]

Proposed weak form of the problem: Find \( u_h \in V_h \) and \( \sigma_h \in S_h \) such that:

\[k(\nabla u_h, \nabla v_h) - \langle \sigma_h \cdot n, v_h \rangle_{\Gamma} + \frac{1}{n}(\nabla v_h, \sigma_h) - \frac{1}{n}k(\nabla v_h, \nabla u_h) = \langle f, v_h \rangle_{\Omega}, \quad \forall v_h \in V_h\]
\[-\frac{1}{nk}(\tau_h, \sigma_h) + \frac{1}{n}(\tau_h, \nabla u_h) - \langle \tau_h \cdot n, u_h \rangle_{\Gamma} = -\langle \tau_h \cdot n, \bar{u} \rangle_{\Gamma}, \quad \forall \tau_h \in S_h\]

where \( n \) is a free parameter. \( S_h \) is elementwise discontinuous.
Suppose that $V_h$ and $S_h$ are such that the following conditions hold for all the elements cut by the boundary $\Gamma$:

$$\forall v_h \in V_h \quad \exists \tau_h \in S_h \mid \|v_h\|_{L^2(\Gamma)}^2 \lesssim \langle \tau_h \cdot n, v_h \rangle_{\Gamma} + \delta_0 h \|\nabla v_h\|_2,$$  \hspace{1cm} (1)

$$\|\tau_h\|_{L^2(\Gamma)} = \|v_h\|_{L^2(\Gamma)} \text{ and } \|\tau_h\|_{L^2(K)}^2 \lesssim h \|v_h\|_{L^2(\Gamma)}^2,$$  \hspace{1cm} (2)

Theorem (Stability)

If the previous hypothesis holds, the problem is stable for any $n > 1$.

In the case of equal interpolation for $V_h$ and $S_h$:

$$\tau_h = n v_h$$

In the case of a linear interpolation for $V_h$ and piecewise constant interpolation for $S_h$ the hypothesis also holds.
Dirichlet boundary conditions in embedded grids
Weak imposition of boundary conditions

Implementation and comparison to Nitche’s method

After condensing the element-dicontinuous stress unknowns:

$$[(1 - \frac{1}{n})K_{uu} - (G_{u\sigma} + K_{u\sigma})K_{\sigma\sigma}^{-1}(K_{\sigma u} + G_{\sigma u})] \cdot U = [f - (G_{u\sigma} + K_{u\sigma})K_{\sigma\sigma}^{-1}g_{\sigma\bar{u}}]$$

We can also write Nitsche’s method in matrix form:

$$[K_{uu} + G_{uu}^1 + G_{uu}^2 + G_{uu}^\alpha] \cdot U = [f + g_{\bar{u}u} + g_{\bar{u}\bar{u}}^\alpha]$$

Comparing both methods we realize that if $S_h$ is rich enough:

Conclusion

The only difference between the presented method and Nitsche’s method is that we have replaced $G_{uu}^\alpha \cdot U$ and $g_{\bar{u}u}^\alpha$ with $-G_{u\sigma}K_{\sigma\sigma}^{-1}G_{\sigma u} \cdot U$ and $-G_{u\sigma}K_{\sigma\sigma}^{-1}g_{\sigma\bar{u}}$. But we do not need any large penalty parameter!
Introducing convection

Two field formulation for the convection-diffusion equation:

\[-k \Delta u + a \cdot \nabla u = f \text{ in } \Omega\]

\[\frac{1}{k} \sigma = \nabla u \text{ in } \Omega\]

Proposed weak form of the problem: Find \(u_h \in V_h\) and \(\sigma_h \in S_h\) such that:

\[
(\nabla u_h, \nabla v_h) - \langle \sigma_h \cdot n, v_h \rangle_{\Gamma} + (v_h, a \cdot \nabla u) + \frac{1}{n} (\nabla v_h, \sigma_h)
\]

\[-\frac{1}{n} k (\nabla v_h, \nabla u_h) + \frac{1}{2} \langle av_h, u_h \rangle = \langle f, v_h \rangle_{\Omega} + \frac{1}{2} \langle av_h, \bar{u}_h \rangle, \quad \forall v_h \in V_h\]

\[-\frac{1}{nk} (\tau_h, \sigma_h) + \frac{1}{n} (\tau_h, \nabla u_h) - \langle \tau_h \cdot n, u_h \rangle_{\Gamma} = -\langle \tau_h \cdot n, \bar{u} \rangle_{\Gamma}, \quad \forall \tau_h \in S_h\]

where

\[a = -a \cdot n, \quad \text{if } a \cdot n < 0\]

\[a = 0, \quad \text{otherwise}\]
Extension to the Stokes problem

Let us consider the three-field formulation for the Stokes problem:

\[-\nu \Delta u + \nabla p = f \quad \text{in } \Omega\]
\[\nabla \cdot u = 0 \quad \text{in } \Omega\]
\[\frac{1}{\nu} \sigma = \nabla u \quad \text{in } \Omega\]

Weak form: Find \( u_h \in V_h, p_h \in Q_h \) and \( \sigma_h \in S_h \) such that:

\[
(1 - \frac{1}{n})\nu(\nabla u_h, \nabla v_h) - (\nabla \cdot v_h, p_h) - \sum \tau_K(\nu \Delta v_h, \nu \Delta u_h + \nabla p_h)K
\]
\[-\langle \sigma_h \cdot n, v_h \rangle \Gamma + \langle n \cdot v_h, p_h \rangle \Gamma + \frac{1}{n}(\nabla v_h, \sigma_h) = \langle f, v_h \rangle \Omega + \sum \tau_K(\nu \Delta v_h, f)_K, \quad \forall v_h
\]
\[-(q_h, \nabla \cdot u_h) + \sum \tau_K(\nabla q_h, \nu \Delta u_h - \nabla p_h)_K + \langle q_h, n \cdot u_h \rangle \Gamma
\]
\[= - \sum \tau_K(\nabla q_h, f)_K + \langle q_h, n \cdot \bar{u} \rangle \Gamma, \quad \forall q_h
\]
\[-\frac{1}{n\nu}(\tau_h, \sigma_h) + \frac{1}{n}(\tau_h, \nabla u_h) - \langle \tau_h \cdot n, u_h \rangle \Gamma = -\langle \tau_h \cdot n, \bar{u} \rangle \Gamma, \quad \forall \tau_h\]
Convection-diffusion problem
Quadratic convergence except for the pure transport equation.

Stokes problem
Numerical examples
Conclusions

- A method for weakly imposing Dirichlet boundary conditions in embedded grids has been proposed.
- The method does not need large penalty parameters in order to ensure stability.
- A symmetric variational form is obtained for symmetric problems.
- Quadratic convergence for linear elements has been attained for the Poisson problem, convection-diffusion equation and the Stokes problem. For the pure transport equation, convergence was linear for linear elements.
- The method is robust and suitable for flow problems, and does not need of additional degrees of freedom.
Outline

1 Introduction

2 Dirichlet boundary conditions in embedded grids
   - Strong imposition of boundary conditions
   - Weak imposition of boundary conditions

3 The Fixed-Mesh ALE method applied to Fluid-Structure Interaction problems

4 Conclusions
Our objectives are:

- To solve FSI problems in moving domains.
- To use a single finite element mesh that covers all the region occupied by the fluid and the structure along its evolution.

To achieve this, we propose a method in which only a background mesh is needed for the whole region to be simulated. The method will be applied to FSI problems and free surface flows.
Suppose we want to solve a problem of the type:
Find a velocity $u$ and a density $\rho$ such that

$$
\rho \left[ \partial_t u + (u - u_m) \cdot \nabla u \right] - \nabla \cdot \sigma = f
$$
$$
\partial_t \rho + (u - u_m) \cdot \nabla \rho + \rho \nabla \cdot u = 0
$$

in a domain $\Omega(t) \subset \Omega^0 \subset \mathbb{R}^d$. Constitutive and state equations need to be appended to this problem. The velocity of the ALE system is $u_m$. We define $\Gamma_{\text{free}}(t)$ as:

$$
\Gamma_{\text{free}}(t) = \partial \Omega(t) \setminus (\partial \Omega^0 \cap \partial \Omega(t))
$$
Suppose $\Omega^0$ is meshed with a finite element mesh $M^0$ and that at time $t^n$ the domain $\Omega^n$ is meshed with a finite element mesh $M^n$. Let $u^n$ be the velocity already computed on $\Omega^n$.

- Define $\Gamma_{\text{free}}^{n+1}$ by updating a boundary function.
- Deform virtually the mesh $M^n$ to $M_{\text{virt}}^{n+1}$ using the classical ALE concepts and compute the mesh velocity $u_{m}^{n+1}$.
- Write down the ALE equations on $M_{\text{virt}}^{n+1}$.
- Split the elements of $M^0$ cut by $\Gamma_{\text{free}}^{n+1}$ to define a mesh on $\Omega^{n+1}$, $M^{n+1}$.
- Project the ALE equations from $M_{\text{virt}}^{n+1}$ to $M^{n+1}$.
- Solve the equations on $M^{n+1}$ to compute $u^{n+1}$ and $\rho^{n+1}$. 
Steps in the algorithm

Top left: $M^0$. Top right: $M^n$. Bot. left: $M_{\text{virt}}^{n+1}$. Bot. right: $M^{n+1}$
Step 1. Mesh velocity

- Updating the boundary defines the deformation of the domain from $\Omega^n$ to $\Omega^{n+1}$.

- The mesh velocity on the boundary points can be computed from their position $x^{n+1}$ and $x^n$:
  $$u_m = (x^{n+1} - x^n)/\delta t.$$  

- On the rest of the nodes, $u_m$ can be computed solving $\Delta u_m = 0$. It is also possible to restrict $u_m \neq 0$ to the nodes next to $\Gamma_{\text{free}}^{n+1}$. 

![Diagram of mesh velocities and deformation](image)
The previous procedure defines the domain $\Omega^{n+1}$ and a mesh $M_{\text{virt}}^{n+1}$. The equations to be solved there, using for example the backward Euler scheme, are:

$$
\rho^{n+1} \left[ \frac{1}{\delta t} (u^{n+1} - u^n) + (u^{n+1} - u_m) \cdot \nabla u^{n+1} \right] - \nabla \cdot \sigma^{n+1} = f^{n+1}
$$

$$
\frac{1}{\delta t} (\rho^{n+1} - \rho^n) + (u^{n+1} - u_m) \cdot \nabla \rho^{n+1} + \rho^{n+1} \nabla \cdot u^{n+1} = 0
$$

in $\Omega^{n+1}$ with boundary conditions on $\Gamma_{\text{free}}^{n+1}$. The mesh velocity $u_m$ has been computed before.
Step 3. Splitting of elements

Elements on the background mesh $M^0$ cut by $\Gamma_{\text{free}}^{n+1}$ are split to define a mesh on $M^{n+1}$.

Both $M^{n+1}$ and $M_{\text{virt}}^{n+1}$ are meshes of the domain $\Omega^{n+1}$. $M^{n+1}$ is a minor modification of the background mesh $M^0$. 
The velocity $u^n$ in $M_{\text{virt}}^{n+1}$ is known because its nodal values correspond to those of mesh $M^n$. Let $P^{n+1}$ be the projection from finite element functions from $M_{\text{virt}}^{n+1}$ to $M^{n+1}$. Denoting $u^{n+1} \equiv P^{n+1}(u^{n+1})$ and $\rho^{n+1} \equiv P^{n+1}(\rho^{n+1})$, the flow equations on $M^{n+1}$ are

$$
\frac{1}{\delta t} \left( \rho^{n+1} - P^{n+1}(\rho^n) \right) + (u^{n+1} - P^{n+1}(u_m)) \cdot \nabla u^{n+1} + \rho^{n+1} \nabla \cdot u^{n+1} = 0
$$

in $\Omega^{n+1}$ with boundary conditions on $\Gamma_{\text{free}}^{n+1}$. The only difficulty is to compute $P^{n+1}(\cdot)$.
Immersed boundary methods (BC’s: through Dirac-type body forces)
Fictitious domain methods (BC’s: through Lagrange multipliers)

We question the treatment of “newly created nodes”. Taylor expansions on which difference approximations are based do not apply there.
Given a position of the fluid front on the fixed mesh, elements cut by the front are split into subelements (only for integration purposes), so that the front coincides with the edges of the subelements.

After deforming the mesh from one time step to the other using classical ALE procedures, results are projected back to the original mesh.

The front is represented by a boundary function, and not by the position of the material points.

Approximate BC’s usually required.
Find a velocity $\mathbf{u}$ such that

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \rho (\mathbf{u} - \mathbf{u}_{\text{dom}}) \cdot \nabla \mathbf{u} = \nabla \cdot \sigma + \mathbf{b},$$

$$\rho J = \rho_0,$$

where,

$$F = \frac{\partial \mathbf{x}}{\partial \mathbf{X}},$$

$$J = \det(F).$$

in a domain $\Omega(t) \subset \Omega^0 \subset \mathbb{R}^d$.

- The mass balance equation is treated in a Lagrangian way.
- Velocity and/or traction boundary conditions must be appended to the problem.
Suppose we are given

\[
\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{u} - \mathbf{u}_{\text{dom}}) \cdot \nabla \mathbf{v} = g(\mathbf{v}),
\]

and that \( \mathbf{v}^n \) is known. \( \mathbf{v}^{n+1} \) can now be found as the solution to the problem:

\[
\delta_t \mathbf{v}^{n+1} \bigg|_{x^n} + (\mathbf{u}^{n+1} - \mathbf{u}_{\text{dom}}^{n+1}) \cdot \nabla \mathbf{v}^{n+1} = g(\mathbf{v}^{n+1}),
\]

where now \( \delta_t \mathbf{v}^{n+1} \bigg|_{x^n} = (\mathbf{v}^{n+1}(\mathbf{x}) - \mathbf{v}^n(\mathbf{x}^n))/\delta t \), being \( \mathbf{x} = \chi_{tn+1,tn}(\mathbf{x}^n) \) the spatial coordinates in \( \Omega(t^{n+1}) \).
This equation can be divided into:

- Material phase:
  \[
  \frac{v^{n+1}(x_{\text{mat}}) - v^n(x^n)}{\delta t} = g(v^{n+1}(x_{\text{mat}})),
  \]
  where \( x_{\text{mat}} = X_{t^{n+1},n}(x^n) \) is the mapping given by the motion of the particles.

- Convective phase:
  \[
  \frac{v^{n+1}(x) - v^{n+1}(x_{\text{mat}})}{\delta t} + (u^{n+1}(x) - u_{\text{dom}}^{n+1}(x)) \cdot \nabla v^{n+1}(x) = 0,
  \]
  where \( x = X_{t^{n+1},n}(x^n) \) are the spatial coordinates in \( \Omega(t^{n+1}) \).

The splitting error is of \( O(\delta t) \).

**Remark** The material phase corresponds to a classical Updated Lagrangian formulation.

**Remark** The FM-ALE method can now be applied to the solid formulation.
Numerical example 1

- Cantilever subject to (horizontal) body forces.
- Finite deformation Neo-Hookean material.
Each problem (fluid and solid) can now be solved using the FM-ALE method.

\[ \Omega_0^0 = \Omega^s(t) \cup \Omega^f(t) \quad \text{and} \quad \Omega_s(t) \cap \Omega_f(t) = \emptyset \]

As a consequence we can divide \( M^0 \) into \( M^t_s \) and \( M^t_f \) such that:

\[ M^0 = M^t_s \cup M^t_f \]

This allows for the use of a single mesh \( M^0 \) to solve both the fluid and the solid mechanics problems in \( t \in [0, T] \).

- Degrees of freedom need to be duplicated in the elements cut by \( \Gamma_{\text{free}} \)
- Coupling conditions can be imposed in an approximate way
Numerical example 2

Thin beam attached to a circular rigid body
Numerical example 3

Rigid bodies falling into water
When dealing with FSI problems we will need to intersect the solid body with the background mesh.

**Algorithmic ingredients**

- Bins search strategy
- Face face intersection
- Subelement integration
Problem data

1. $Re \sim 65000$
2. Particular boundary conditions
3. Surface tension is not considered
4. LES Smagorinsky model for turbulent flows
5. Heavy preconditioner (sparsity ratio $\sim 5$)
6. 30 – 200 GMRES iterations
Fixed-Mesh ALE method

The water entry of a decelerating sphere

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FM-ALE for FSI problems
Fixed-Mesh ALE method

The water entry of a decelerating sphere
The water entry of a decelerating sphere

Experimental results

Numerical 2469600 elements
Numerical 514000 elements

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The water entry of a decelerating sphere
Conclusions

The FM-ALE method avoids to use different meshes along the calculation.

The additional cost is due to

- The representation of the moving surface (e.g. by a level set function).
- The projection of nodes of the deformed mesh onto the background mesh.

The benefit is that

- Meshing the background domain is easy.
- Mesh distortion does not occur.
- The controversial issue of newly created nodes disappears.
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Conclusions

We have proposed two methods for imposing Dirichlet boundary conditions in embedded grids:

**Strong imposition of boundary conditions**


**Weak imposition of boundary conditions**

We have presented the FM-ALE method for multiphysics problems in domains undergoing large deformations:

The FM-ALE applied to multiphysics problems


THANK YOU!
The Fixed-Mesh ALE method for Fluid-Structure Interaction problems

Ramon Codina and Joan Baiges

Technical University of Catalonia

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